Valerii Vinokur


Superconducting hybrid nanostructures: physics and application
Quantum mechanics in $n$-dimensions is equivalent to statistical physics in $(n+1)$-dimensions.

$z \leftrightarrow it$

(Wick rotation)
Field Theory and Statistical Mechanics are closely connected. A Wick rotation $t \rightarrow i/(kT)$ will take you from one to the other.

Quantum Mechanics and Classical Mechanics are closely connected. Both employ Hamiltonians as basic generators of time development as do Field Theory and Statistical Mechanics.

All four have a dual structure in which terms in the Hamiltonian both describe measurable quantities and equally generate changes in development.

All four have the same structure: Poisson Bracket and Commutator, conjugate variables = p’s and q’s.
Vector boson exchange model as the theoretical explanation of the strong nuclear force in 1935 (Yukawa)
In 1938 Stueckelberg recognized that massive electrodynamics contains a hidden scalar, and formulated an affine version of what would become known as the Abelian Higgs mechanism
proposed the law of conservation of baryon number; In 1952 he proved the principle of semi-detailed balance for kinetics without microscopic reversibility.
In 1953 he and the mathematician André Petermann discovered the renormalization group. [10]
Free energy and partition function of vortices

\[ F_N = \int dz \left[ \frac{\varepsilon_1}{2} \sum_i \left( \frac{dr_i}{dz} \right)^2 + U(r_i) + \sum_{i<j} V(r_{ij}) \right] \]

- Linear tension
- Elastic energy of a single vortex
- Potential energy of a single vortex
- Vortex-vortex interaction

\[ Z = \int Du \exp(-F_N/T) \]
Vortex-particle duality

\[ \exp \left( -\frac{1}{T} \int dz \frac{\varepsilon_l}{2} \left( \frac{dr}{dz} \right)^2 \right) \]

\[ \exp \left( \frac{i}{\hbar} \int dt \frac{m}{2} \left( \frac{dr}{dt} \right)^2 \right) \]

statistical physics of vortices in \((n+1)\)-dimensions is equivalent to quantum mechanics of particles in \(n\)-dimensions
Vortex-particle duality

QM$\leftrightarrow$SP mapping in the presence of pinning: vortex – quantum particles mapping

D. R. Nelson and V. M. Vinokur
*Boson localization and correlated pinning of superconducting vortex arrays, PRB 48, 13 060 (1993)*

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**TABLE I.** Boson analogy applied to vortex transport.

<table>
<thead>
<tr>
<th>Charged bosons</th>
<th>Mass</th>
<th>$\hbar$</th>
<th>$\beta\hbar$</th>
<th>Pair potential</th>
<th>Charge</th>
<th>Electric field</th>
<th>Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superconducting</td>
<td>$\varepsilon_1$</td>
<td>$T$</td>
<td>$L$</td>
<td>$2\varepsilon_0 K_0(r/\lambda_{ab})$</td>
<td>$\phi_0$</td>
<td>$\frac{\hat{z} \times \mathbf{J}}{c}$</td>
<td>$\mathcal{E}$</td>
</tr>
<tr>
<td>Vortices</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Non-Hermitian quantum mechanics
Non-Hermitian quantum mechanics

We treat vortex transport in the Bose-glass and Mott-insulator phases by adapting the equilibrium analysis presented in Secs. II and III. When $\mathbf{J} \perp \mathbf{B}$ a Lorentz force per unit length perpendicular to $\mathbf{J}$ acts on all vortices,\(^{64}\)

$$f_L = \frac{\phi_0}{c} \hat{z} \times \mathbf{J},$$

which leads to an additional term

$$\delta F_N = - f_L \cdot \sum_{j=1}^{N} \int_0^L r_j(z) dz$$

in the vortex-free energy Eq. (2.1a). We exploit the analogy with boson quantum mechanics by noting that this term represents a fictitious “electric field” $\mathbf{E} = (1/c) \hat{z} \times \mathbf{J}$ acting on particles with charge $\phi_0$.\(^{65}\)
Non-Hermitian quantum mechanics

\[ \mathbf{a}_{\text{mic}}(\mathbf{r}) = \frac{J_z}{2} (\mathbf{\hat{z}} \times \mathbf{r}) \]

“magnetic field” \( \sim i(\nabla \times \mathbf{a}_{\text{mic}}) \) is purely imaginary.

FIG. 9. Effective flux-line potential near a columnar pin with a current \( J \parallel \mathbf{B} \). The vortex bound state is unstable; the equivalent quantum problem is \( \beta \) decay.
Hermiticity of the quantum mechanics

The operator $\hat{A}^\dagger$ is called the hermitian conjugate of $\hat{A}$ if

$$\int (\hat{A}^\dagger \psi)^* \psi \, dx = \int \psi^* \hat{A} \psi \, dx$$

The operator $\hat{A}$ is called hermitian if

$$\int (\hat{A} \psi)^* \psi \, dx = \int \psi^* \hat{A} \psi \, dx$$

The eigenvalues of hermitian operators are real.

Since we expect that the average values of operators representing observable quantities are real, then these operators are obliged to be Hermitian (or self-adjointed).
Um diese Schwierigkeit zu überwinden, müssen wir annehmen, daß die Schwingungen gedämpft sind, und $\mathbf{E}$ komplex setzen:

$$E = E_0 + i \frac{\hbar \lambda}{4 \pi},$$

wo $E_0$ die gewöhnliche Energie ist und $\lambda$ das Dämpfungsdekrement (Zerfallskonstante). Dann sehen wir aber aus den Relationen (2a) und (2b),
Proof

Let $\psi$ be an eigenfunction of $\hat{A}$ with eigenvalue $a$:

$$\hat{A}\psi = a\psi$$

then we have

$$\int (\hat{A}\psi)^* \psi \, dx = \int (a\psi)^* \psi \, dx = a^* \int \psi^* \psi \, dx$$

and by hermiticity of $\hat{A}$ we also have

$$\int (\hat{A}\psi)^* \psi \, dx = \int \psi^* \hat{A}\psi \, dx = a \int \psi^* \psi \, dx$$

$$(a^* - a) \int \psi^* \psi \, dx = 0$$

and since $\int \psi^* \psi \, dx \neq 0$, we get

$$a^* - a = 0$$

How’s about the converse theorem?
Ordinarily, one imposes the condition $H^\dagger = H$ on the Hamiltonian. However, replacing this mathematical condition by the weaker and more physical requirement

$$H^{\mathcal{PT}} = H$$

where $\mathcal{PT}$ represents combined parity reflection and time reversal $\mathcal{PT}$, one obtains new classes of complex Hamiltonians whose spectra are still real and positive.
The postulate of Hermiticity of the Hamiltonian has been an integral component of quantum mechanics ever since its inception.

 Nonetheless, while the eigenvalues of a Hermitian Hamiltonian are all real, Hermiticity of a Hamiltonian is only a sufficient condition for such reality.

\[ M = \begin{pmatrix} 1 + i & s \\ s & 1 - i \end{pmatrix} \]

\[ E_{\pm} = 1 \pm (s^2 - 1)^{1/2} \]

both of these eigenvalues are real if \( s \) is real and greater than one
PT-symmetric Quantum Theory

\[ H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real}) \]

The spectrum of \( H \) for \( \varepsilon > 0 \) is real

When \( \varepsilon < 0 \), the \( PT \) symmetry of \( H \) is broken

Energy levels of the parametric family of the above Hamiltonians. When \( \varepsilon \geq 0 \) the eigenvalues are all real and positive and they increase with increasing \( \varepsilon \). When \( \varepsilon \) decreases below 0, the eigenvalues disappear into the complex plane as complex conjugate pairs. One real eigenvalue remains when \( \varepsilon \) is less than approximately \(-0.57\), and as \( \varepsilon \to -1 \) from above, this eigenvalue becomes infinite.

At \( \varepsilon = 1 \) and at \( \varepsilon = 2 \)

\[ H = p^2 + ix^3 \quad \text{and} \quad H = p^2 - x^4 \]

The eigenvalues of these Hamiltonians are all real, positive and discrete.

PT-symmetric Quantum Theory


Broken and unbroken PT symmetry

For the case of a $PT$-symmetric Hamiltonian, the $PT$ operator commutes with the Hamiltonian $H$

assume that an eigenstate $\psi$ of the Hamiltonian $H$ is also an eigenstate of the $PT$ operator.

$PT\psi = \lambda\psi$

Multiply this by $PT$ on the left and use

$(PT)^2 = 1$

$H\psi = E\psi$

$\psi = (PT)\lambda(PT)^2\psi$

Since $T$ is antilinear

$\psi = \lambda^*\lambda\psi = |\lambda|^2\psi$

$|\lambda|^2 = 1$

$\lambda = e^{i\alpha}$

Next, multiply the eigenvalue equation by $PT$ on the left

$(PT)H\psi = (PT)E(PT)^2\psi$

$H\lambda\psi = (PT)E(PT)\lambda\psi$

Use that $T$ is antilinear and obtain:

$E\lambda\psi = E^*\lambda\psi$

$E = E^*$

If every eigenfunction of a $PT$–symmetric Hamiltonian is also an eigenfunction of the $PT$ operator, we say that the $PT$ symmetry of $H$ is unbroken.
Boundary conditions for Schrödinger equation

\[- \psi''(x) + x^2 (ix)^\epsilon \psi(x) = E \psi(x)\]

WKB approximation

\[-y''(x) + V(x)y(x) = 0\]

\[y(x) \sim \exp \left[ \pm \int^x ds \sqrt{V(s)} \right]\]

\[|x| \to \infty\]

Stokes wedges in the complex-\(x\) plane containing the contour on which the eigenvalue problem for the differential equation for \(\epsilon = 2.2\) is posed. In these wedges \(\psi(x)\) vanishes exponentially as \(|x| \to \infty\). The eigenfunction \(\psi(x)\) vanishes most rapidly at the centres of the wedges.
PT-symmetric quantum mechanics

Reminder: conventional QM

(a) Eigenfunctions and eigenvalues of $H$.

(b) Orthogonality of eigenfunctions

\[
(\psi, \phi) \equiv \int dx \, [\psi(x)]^* \phi(x)
\]

\[
(\psi_m, \phi_n) = 0
\]

(c) Orthonormality of eigenfunctions.

\[
(\psi_n, \psi_n) = 1
\]

(d) Completeness of eigenfunctions.

\[
\chi = \sum_{n=0}^\infty a_n \psi_n \quad \sum_{n=0}^\infty [\psi_n(x)]^* \psi_n(y) = \delta(x - y)
\]

(e) Reconstruction of the Hamiltonian $H$ and Green’s function $G$, and calculation of the spectral Zeta function.

\[
\sum_{n=0}^\infty [\psi_n(x)]^* \psi_n(y) E_n = H(x, y)
\]

\[
\sum_{n=0}^\infty [\psi_n(x)]^* \psi_n(y) \frac{1}{E_n} = G(x, y)
\]

\[
\int dx \, G(x, x) = \sum_{n=0}^\infty \frac{1}{E_n}
\]
PT-symmetric quantum mechanics

(f) Time evolution and unitarity.

\[ \langle \chi(t), \chi(t) \rangle = \langle \chi(0)e^{iHt}, e^{-iHt}\chi(t) \rangle = \langle \chi(0), \chi(0) \rangle \]

(g) Observables. An observable is represented by a linear Hermitian operator.
**PT-symmetric quantum mechanics**

(a) *Eigenfunctions and eigenvalues of $H$. Eigenvalues are all real provided that the PT symmetry of $H$ is unbroken*

(b) *Orthogonality of eigenfunctions.*

\[
(\psi, \phi) \equiv \int_C dx \, [\psi(x)]^{PT} \phi(x) = \int_C dx \, [\psi(-x)]^* \phi(x)
\]

*C is a contour in the Stokes wedges*
(c) **The CPT inner product.**

To construct an inner product with a positive norm for a complex non-Hermitian Hamiltonian having an *unbroken PT* symmetry, we will construct a new linear operator $C$ that commutes with both $H$ and $PT$. The properties of $C$ are similar to those of the charge conjugation operator in particle physics.

\[
\langle \psi | \chi \rangle_{CPT} = \int dx \, \psi^{CPT}(x) \chi(x)
\]

\[
\psi^{CPT}(x) = \int dy \, C(x, y) \psi^*(-y)
\]
PT-symmetric quantum mechanics

(d) PT-symmetric normalization of the eigenfunctions and the statement of completeness.

The eigenfunctions $\psi_n(x)$ of $H$ are also eigenfunctions of the $PT$ operator with eigenvalue $\lambda = \exp(i\langle \rangle)$, where $\lambda$ and $\langle \rangle$ depend on $n$. Thus, we can construct $PT$-normalized eigenfunctions $\varphi_n(x)$ defined by

$$\varphi_n(x) \equiv e^{-i\alpha/2} \psi_n(x)$$

By this construction, $\varphi_n(x)$ is still an eigenfunction of $H$ and it is also an eigenfunction of $PT$ with eigenvalue 1. One can also show both that the algebraic sign of the $PT$ norm of $\varphi_n(x)$ is $(-1)^n$ for all $n$ and for all values of $\epsilon > 0$.

$$\int_C dx \left[ \varphi_n(x) \right]^{PT} \varphi_n(x) = \int_C dx \left[ \varphi_n(-x) \right]^* \varphi_n(x) = (-1)^n$$

Completeness:

$$\sum_{n=0}^{\infty} (-1)^n \varphi_n(x) \varphi_n(y) = \delta(x - y)$$

$\epsilon > 0$
(e) **Coordinate-space representation of $H$ and $G$ and the spectral Zeta function.** From the statement of completeness we can construct coordinate-space representations of the linear operators. For example, since the coordinate-space representation of the parity operator is $P(x, y) = \delta(x + y)$, we have

\[
P(x, y) = \sum_{n=0}^{\infty} (-1)^n \phi_n(x)\phi_n(-y)
\]

\[
H(x, y) = \sum_{n=0}^{\infty} (-1)^n E_n \phi_n(x)\phi_n(y)
\]

\[
G(x, y) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{E_n} \phi_n(x)\phi_n(y)
\]

\[
\int dy \, H(x, y)G(y, z) = \delta(x - z)
\]
PT-symmetric quantum mechanics

\[ H = \hat{p}^2 + \hat{x}^2 (ix)^\epsilon \]

\[ \left( -\frac{d^2}{dx^2} + x^2 (ix)^\epsilon \right) G(x, y) = \delta(x - y) \]

\[
\sum_n \frac{1}{E_n} = 1 + \frac{\cos \left( \frac{3\epsilon \pi}{2\epsilon + 8} \right) \sin \left( \frac{\pi}{4 + \epsilon} \right)}{\cos \left( \frac{\epsilon \pi}{4 + 2\epsilon} \right) \sin \left( \frac{3\pi}{4 + \epsilon} \right)} \frac{\Gamma \left( \frac{1}{4 + \epsilon} \right) \Gamma \left( \frac{2}{4 + \epsilon} \right) \Gamma \left( \frac{\epsilon}{4 + \epsilon} \right)}{(4 + \epsilon)^{\frac{4 + 2\epsilon}{4 + \epsilon}} \Gamma \left( \frac{1 + \epsilon}{4 + \epsilon} \right) \Gamma \left( \frac{2 + \epsilon}{4 + \epsilon} \right)}
\]
**PT-symmetric quantum mechanics**

(f) *Construction of the C operator.*

\[
C(x, y) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)
\]

\[
\int dy \, C(x, y)C(y, z) = \delta(x - z)
\]

Thus, the eigenvalues of \( C \) are \( \pm 1 \).

\( C \) commutes with \( H \). Therefore, since \( C \) is linear, the eigenstates of \( H \) have definite values of \( C \). Specifically,

\[
C \phi_n(x) = \int dy \, C(x, y)\phi_n(y)
\]

\[
= \sum_{m=0}^{\infty} \phi_m(x) \int dy \, \phi_m(y)\phi_n(y) = (-1)^n \phi_n(x)
\]

\[
\mathcal{P}^2 = C^2 = 1, \text{ but } \mathcal{P} \neq C \text{ because } \mathcal{P} \text{ is real, while } C \text{ is complex.}
\]

However, \( C \) does commute with \( PT \).
PT-symmetric quantum mechanics

(g) Positive norm and unitarity in PT-symmetric quantum mechanics. Having constructed the operator $C$, we can now use the new $CPT$ inner product. Like the $PT$ inner product, this new inner product is phase independent. Also, because the time evolution operator (as in ordinary quantum mechanics) is $\exp(-iHt)$ and because $H$ commutes with $PT$ and with $CPT$, both the $PT$ inner product and the $CPT$ inner product remain time independent as the states evolve. However, unlike the $PT$ inner product, the $CPT$ inner product is positive definite because $C$ contributes a factor of $-1$ when it acts on states with negative $PT$ norm. In terms of the $CPT$ conjugate, the completeness condition reads

$$
\sum_{n=0}^{\infty} \phi_n(x)[CPT\phi_n(y)] = \delta(x - y)
$$

**Unitarity:**

$$
\psi_t(x) = e^{-iHt} \psi_0(x)
$$

With respect to the $CPT$ inner product the norm of $\psi_t(x)$ does not change in time because $H$ commutes with $CPT$. 
**Observables**

In ordinary quantum mechanics the condition for a linear operator $A$ to be an observable is that $A = A^\dagger$. This condition guarantees that the expectation value of $A$ in a state is real. Operators in the Heisenberg picture evolve in time according to $A(t) = e^{iHt}A(0)e^{-iHt}$, so this Hermiticity condition is maintained in time.

In $PT$-symmetric quantum mechanics the equivalent condition is that at time $t = 0$ the operator $A$ must obey the condition $A^T = CPT(A)CPT$, where $A^T$ is the transpose of $A$. If this condition holds at $t = 0$, then it will continue to hold for all time because we have assumed that $H$ is symmetric ($H = H^T$). This condition also guarantees that the expectation value of $A$ in any state is real.

Operators $C$ and $H$ are observable.

However, the $\hat{x}$ and $\hat{p}$ operators are not.
Applications

Observation of parity-time symmetry in optics

Christian E. Rüter¹, Konstantinos G. Makris², Ramy El-Ganainy², Demetrios N. Christodoulides², Mordechai Segev³ and Detlef Kip¹*

\[ i \frac{\partial E}{\partial z} + \frac{1}{2k} \frac{\partial^2 E}{\partial x^2} + k_0 [n_R(x) + i n_I(x)] E = 0 \]

where \( k_0 = 2\pi/\lambda \), \( k = k_0 n_0 \), \( \lambda \) is the wavelength of light in vacuum and \( n_0 \) represents the substrate index.

Parity-time-symmetric whispering-gallery microcavities

Bo Peng¹⁺, Şahin Kaya Özdemir¹⁺⁺, Fuchuan Lei¹,², Faraz Monifi¹, Mariagiovanna Gianfreda³,⁴, Gui Lu Long²,⁵, Shanhui Fan⁶, Franco Nori⁷,⁸, Carl M. Bender³ and Lan Yang¹*

E is the electric field
Applications: Nonequilibrium dissipative systems

Dynamics: gain and loss balance

Suppose that we have a system transmitting something (water, energy, etc …)

PT-symmetric state implies the exact balance between the gain and loss: the steady-state dynamics. If PT symmetry is broken, the system transits into a non-stationary state.

Applications: \( PT \) symmetry in superconductivity


A finite superconducting wire subjected to an electric current that is fed into one of its ends, creating a voltage difference across the wire.

\[
\psi_t + i \varphi \psi = \psi_{xx} + \Gamma \psi - |\psi|^2 \psi.
\]

Conservation of the current \( I \) implies the relation

\[
\frac{i}{2} (\psi \psi^* - \psi_x \psi^*_x) - \sigma \varphi_x = I
\]
Applications: \( PT \) symmetry in superconductivity

N. M. Chtchelkatchev, A. A. Golubov, T. I. Baturina, and V. M. Vinokur, Stimulation of the Fluctuation Superconductivity by PT Symmetry

\[ \mathcal{H}_{\text{eff}} = -D \nabla^2_x - \tau^{-1} - 2i \varphi \]

\[ \delta R \sim -\frac{E_{\text{Th}}^2 d}{e^2 k_B T L} \frac{1}{\sqrt{[\tau^{-1} - \text{Re}\varepsilon_0(\varepsilon)/\hbar]^2 + [\Gamma(\varepsilon) + \gamma]^2}} \]
Applications: *Dynamic Mott transition*


There are materials that should conduct electricity under conventional band theories, but are insulators due to electron–electron interactions localizing electrons.

**DISCUSSION OF THE PAPER BY DE BOER AND VERWEY**

Reported by N. F. MOTT

with the help of some notes from R. PEIERLS

---

*Weak interactions: superfluidity*

*Strong interactions: Mott insulator which preserves all lattice symmetries*
How does the Mott gap collapse under the applied drive?

Mott gap: $\Delta = E_1 - E_0$

the transition probability $P(|0\rangle \rightarrow |1\rangle) \sim e^{-2\gamma}$

A. M. Dykhne


Landau-Dykhne formula

$$\gamma = \text{Im} \int_{-\infty}^{\infty} dt \left[ E_1(\Phi(t)) - E_0(\Phi(t)) \right]$$

$\Phi$ is the external potential.
How does the Mott gap collapse under the applied drive?

\[ \gamma = \text{Im} \int_{-\infty}^{\infty} dt \left[ E_1(\Phi(t)) - E_0(\Phi(t)) \right] \]

Add the *imaginary* field $iF$ to the Hamiltonian describing Mott insulator. The resulting non-Hermitian Hamiltonian is $PT$-symmetric. At some threshold field $F_{th}$, the $PT$ symmetry will be lost and the ground and the first excited levels merge.

This will provide the description of the collapse of the Mott gap.
How does the Mott gap collapse under the applied drive?

\[ P = e^{-2\gamma} \]

\[
\exp \left\{ -\frac{2}{\hbar} \Im \int_{t_i}^{T_c} \left[ E_1(\Phi(t)) - E_0(\Phi(t)) \right] dt \right\}
\]

The integral over time can be replaced with the integral over the potential

\[ \Phi_c = Ft \pm i\chi_c \]

\( \chi(F) \) is a phenomenological gauge field, \( \chi(0)=0, \chi_c=\chi(F_c) \)

\[ \gamma = \frac{1}{F} \Re \int_{\chi_i}^{\chi_c} d\chi \left[ E_1(\chi) - E_0(\chi) \right] \]

\[ \chi_i \ll \chi_c, \quad \gamma \sim \Delta^2 / vF \]

\[ v = \left| \frac{d\Delta}{dt} \right| / F \]
How does the Mott gap collapse under the applied drive?

\[ S = \int d^2x \, dt \left[ \Psi^\dagger \frac{\partial}{\partial \tau} \Psi + D |\nabla \Psi|^2 + m^2 |\Psi|^2 + u |\Psi|^4 \right] \]

\[ \frac{\partial \Psi}{\partial t} + \eta \frac{\delta H}{\delta \Psi^*} = 0 \]

Linearize the problem and seek for a solution in the form

\[ \Psi(x, y, t) = e^{ik_y y - Et} u(x) \]

\[ H u = E u \]

\[ u_{xx} + ix I u = -E u \]

\[ u(\pm 1) = 0. \]
How does the Mott gap collapse under the applied drive?

The solution to this equation is the combination of the Airy functions.

We can expand near the bifurcation point:

\[ E_1 - E_0 \approx E_T \sqrt{\eta \left(1 - \frac{I^2}{I_0^2}\right)} \sim E_T \sqrt{(I_0 - I)/I_0}, \]

Therefore

\[ \gamma \sim \left(1 - \frac{I}{I_0}\right)^{3/2} \]

Which is exactly the scaling we have found experimentally!!
Dissipation and non-Hermiticity

How to connect imaginary field and a real applied drive?

Consider Legendre transformed Hamiltonian

\[ H' = H - i\lambda J \]

T. Antal et al., Phys. Rev. Lett. 78 (1997);

Real \( \Box \) is equivalent to the imaginary vector potential

At small \( \Box \), the spectral gap is that of \( H \quad \Box \quad \langle J \rangle = 0 \)
(Real eigenvalues for \( H' \))

At large \( \Box \), the eigenstates are essentially those of \( J \quad \Box \quad \langle J \rangle \quad \Box \quad 0 \)
(Complex eigenvalues)
Dissipation and non-Hermiticity

Construct the density matrix for our non-Hermitian model

\[ \rho(t) = \frac{e^{-iH't}\rho(0)e^{-iH't}}{\text{Tr}[e^{-iH't}\rho(0)e^{-iH't}]} \]

\[ \langle A \rangle \equiv \text{Tr}(\rho A) \]

Finite real part of \( \square \) ensures relaxation to the nonequilibrium steady state:

\[ \frac{d\langle J \rangle}{dt} = -\lambda(\langle J^2 \rangle - \langle J \rangle^2) \]

Model then does describe a nonequilibrium (dissipative) transition. Factor \( \square \) characterizes both dissipation & drive.

Below some critical field \( F_c \) (corresponding to a critical \( \square_c \)), the spectral gap is finite and \( \langle J \rangle = 0 \), eigenvalues of \( H' \) are real. At larger fields, the finite current appears, eigenvalues are complex.

Eigenstates lose \( PT \) symmetry.
Small magnetic systems

Slonczewski spin-transfer torque

Action of STT is PT-invariant: $e_p \rightarrow -e_p, t \rightarrow -t$
Small magnetic systems: Landau-Lifshitz equation

\[(1 + \alpha^2)\mathbf{S} = \gamma \mathbf{H}_{eff} \times \mathbf{S} + \frac{\alpha}{S} [\gamma \mathbf{H}_{eff} \times \mathbf{S}] \times \mathbf{S} + \frac{1}{S} [\mathbf{j} \times \mathbf{S}] \times \mathbf{S} - \alpha \mathbf{j} \times \mathbf{S}\]

\(\alpha\) is the damping constant

\[\mathbf{j} = e_p \frac{\hbar}{2e} \eta\]

Is the spin-angular momentum introduced per second

\[\eta = (J_\uparrow - J_\downarrow)/(J_\uparrow + J_\downarrow)\]

\[\gamma \mathbf{H}_{eff} = \nabla_S E(\mathbf{S})\]

\[\gamma = g\mu_B/\hbar\]

is the effective magnetic field

\[E(\mathbf{S})\]

is the magnetic energy
Non-Hermitean spin Hamiltonian

Conjecture: 

\[ H = E(S) + \frac{ij \cdot S}{1 - i\alpha} \]

describes non-equilibrium dynamics of the spin

STT→ ‘imaginary magnetic field’ \( ij \)
Non-Hermitian spin Hamiltonian

\[ H = E(S) + \frac{i j \cdot S}{1 - i \alpha} \]

The equation of motion for a single spin in stereographic projection coordinate \( \zeta \):

\[ \partial_t \zeta = \frac{i}{2S} (1 + |\zeta|^2)^2 \frac{\partial H}{\partial \zeta} \]

In the classical limit of large spin

\[ S \gg 1 \]
**PT symmetry-breaking**

\[ H_{PT} = (D/S)S_z^2 + h_x S_x + i\beta S_y \]

Here D is the uniaxial anisotropy constant, \( \alpha = 0 \)

Symmetry-breaking occurs at \( h_x = 1 \)

\( D = 2.0 \)
symmetry-breaking first occurs in a finite region of initial conditions (dark gray).

\[ H_{PT} = \frac{D}{S} S_z^2 + h_x S_x + i\beta S_y \]

\( \beta = 4.7 \)

\[ \zeta = s_x + i s_y / s_z \]
Linear dynamics of classical spin as Möbius transformation

FIG. 1. Geodesics of an elliptic (a), hyperbolic (b) and loxodromic (c) Möbius transformation, corresponding in spin dynamics to the applied real magnetic field along the x-axis (a), imaginary magnetic field along the y-axis (b) and complex magnetic field along the y-axis (c).
Linear dynamics of classical spin as Möbius transformation

FIG. 2. Transition between elliptic (a) and loxodromic (c) Möbius transformations by increasing $\epsilon$ past the critical value 1, at which one obtains a parabolic transformation (b). In classical spin dynamics this transition corresponds to $PT$ symmetry-breaking.
Thank you for your attention!